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LETTER TO THE EDITOR

Fixation in a cyclic Lotka–Volterra model

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Abstract. We study a cyclic Lotka–Volterra model of N interacting species populating a d-dimensional lattice. In the realm of a Kirkwood approximation, a critical number of species $N_c(d)$ above which the system fixates is determined analytically. We find $N_c = 5, 14, 23$ in dimensions $d = 1, 2, 3$, in remarkably good agreement with simulation results in two dimensions.

A cyclic variant of the Lotka–Volterra model of interacting populations, originally introduced by Vito Volterra for the description of the struggle for existence among species [1, 2], has then appeared in a number of apparently unrelated fields ranging from plasma physics [3] to integrable systems [4, 5]. Recently, the cyclic Lotka–Volterra model (also known as the cyclic voter model) has attracted considerable interest as it was realized that introduction of the spatial structure drastically enriches the dynamics [6–11]. Namely, if species live on a one-dimensional (1D) lattice, a homogeneous initial state evolves into a coarsening mosaic of interacting species. This heterogeneous spatial structure spontaneously develops when the number of species is sufficiently small, $N < N_c$, where $N_c = 5$ in one dimension [6, 8, 11]. For $N \geqslant N_c$ fixation occurs, i.e. the system approaches a frozen state. Little is known in higher dimensions, not even the existence of N_c has yet been established theoretically or numerically (in simulations on two-dimensional (2D) lattices with $N \leqslant 10$ species, no sign of fixation has been found and instead a reactive steady state has been observed [6–11]). In this letter we investigate the cyclic Lotka–Volterra model in the framework of a Kirkwood-like approximation. This approach predicts a *finite* N_c in all spatial dimensions.

In the following, we shall use the language of the voter model [12]. Consider the cyclic voter model with N possible opinions. Each site of a d-dimensional cubic lattice is occupied by a voter which has an opinion labelled by α, with $\alpha = 1, \ldots, N$. Voters can change their opinions in a cyclic manner, $\alpha \to \alpha - 1$ modulo N, according to the opinions of their neighbourhood. Specifically, the following sequential dynamics is implemented: (i) we randomly choose a site (of opinion α, say) and one of its $2d$ nearest neighbours (of opinion β); (ii) if $\beta = \alpha - 1$, then the chosen site changes its opinion from α to $\beta = \alpha - 1$; and (iii) otherwise, opinion does not change. We set the timescale so that in unit time each site of the lattice is chosen once on average. When $N = 2$ the cyclic voter model is

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identical to the classic voter model which is solvable in arbitrary dimension [13]; therefore in the following we assume that $N \geqslant 3$.

In order to simplify the notation, we consider first a 1D chain. We define $p_{\alpha_1,\ldots,\alpha_i}(t)$ as the probability that a randomly chosen segment of i consecutive sites contains opinions $\alpha_1,\ldots,\alpha_i$. For instance, the one-point function $p_\alpha(t)$ is the density of opinion α. It obeys

$$2\frac{\mathrm{d}p_\alpha}{\mathrm{d}t} = p_{\alpha,\alpha+1} + p_{\alpha+1,\alpha} - p_{\alpha,\alpha-1} - p_{\alpha-1,\alpha}. \tag{1}$$

We consider random and uncorrelated initial opinion distributions. This implies $p_\alpha(0) = 1/N$, and generally $p_{\alpha_1,\ldots,\alpha_i}(0) = 1/N^i$. Symmetry leads to $p_{\alpha,\alpha+1} = p_{\alpha+1,\alpha} = p_{\alpha-1,\alpha} = p_{\alpha,\alpha-1}$, so equation (1) gives $\mathrm{d}p_\alpha/\mathrm{d}t = 0$ and hence $p_\alpha(t) = 1/N$. Although the dynamics is non-conserved, i.e. the densities can change locally, we see that for the symmetric initial conditions with equal concentrations the densities are conserved globally. The two-point functions obey

$$2\frac{\mathrm{d}p_{\alpha,\alpha}}{\mathrm{d}t} = -p_{\alpha-1,\alpha,\alpha} - p_{\alpha,\alpha,\alpha-1} + p_{\alpha+1,\alpha} + p_{\alpha,\alpha+1} + 2p_{\alpha,\alpha+1,\alpha} \tag{2}$$

$$2\frac{\mathrm{d}p_{\alpha,\alpha+1}}{\mathrm{d}t} = -p_{\alpha,\alpha+1} - p_{\alpha-1,\alpha,\alpha+1} - p_{\alpha,\alpha+1,\alpha} + p_{\alpha,\alpha+1,\alpha+1} + p_{\alpha,\alpha+2,\alpha+1} \tag{3}$$

which are valid for arbitrary $N \geqslant 3$, and

$$2\frac{\mathrm{d}p_{\alpha,\alpha+i}}{\mathrm{d}t} = -p_{\alpha-1,\alpha,\alpha+i} - p_{\alpha,\alpha+i,\alpha+i-1} + p_{\alpha,\alpha+1,\alpha+i} + p_{\alpha,\alpha+i+1,\alpha+i}. \tag{4}$$

Equations (4) apply for $N \geqslant 4$, $2 \leqslant i \leqslant N-2$. Of course, the indices are taken modulo N. Finally, for symmetry reasons $p_{\alpha,\alpha+N-1} = p_{\alpha,\alpha-1} = p_{\alpha,\alpha+1}$, and more generally $p_{\alpha,\alpha+N-i} = p_{\alpha,\alpha+i}$.

The above equations are exact and normalization can be verified. For instance, $\sum_{1\leqslant i\leqslant N} p_{\alpha,\alpha+i} = p_\alpha = 1/N$. Equations (1)–(4) are the first of an infinite hierarchy of equations which are hardly solvable. However, a considerable insight can be gained within the two-sites mean-field approximation (also called Kirkwood approximation) that expresses k-point functions via one- and two-point functions [14]. For example, the three-point functions read

$$p_{\alpha_1,\alpha_2,\alpha_3} = \frac{p_{\alpha_1,\alpha_2}\,p_{\alpha_2,\alpha_3}}{p_{\alpha_2}}. \tag{5}$$

This kind of factorization approximation originally developed in the realm of equilibrium statistical mechanics has proven to be remarkably successful for a number of non-equilibrium processes as well [15].

The ansatz of equation (5) closures the above rate equations, for example, equation (4) becomes

$$\dot{r}_i = \frac{Nr_1}{2}(r_{i-1} - 2r_i + r_{i+1}) \tag{6}$$

where $r_i = p_{\alpha,\alpha+i}$, so for instance $r_1 = p_{\alpha,\alpha+1}$ is the concentration of reactive pairs. Note that the evolution rules which define the model are translationally invariant in 'opinion space' and therefore for translationally invariant initial distributions the two-point correlator $p_{\alpha,\beta}$ is only a function of $\beta - \alpha$. Hence Nr_i is the probability that opinions of any two randomly chosen consecutive sites differ by i. The normalization condition thus reads $\sum_{0\leqslant i\leqslant N-1} r_i = 1/N$. Upon combining with the symmetry requirement, $r_i = r_{N-i}$, the normalization condition yields

$$r_0 + 2\sum_{i=1}^{M-1} r_i + r_M = \frac{1}{N} \qquad N = 2M \tag{7}$$

$$r_0 + 2\sum_{i=1}^{M} r_i = \frac{1}{N} \qquad N = 2M + 1. \tag{8}$$

We now turn to the arbitrary dimension d. Making use of the compact notation r_i, we arrive at the generalization of the previous rate equations (valid within the realm of the Kirkwood approximation)

$$\begin{aligned} \dot{r}_0 &= \frac{2d-1}{2d} N r_1 \left[\frac{2}{(2d-1)N} - 2r_0 + 2r_1\right] \\ \dot{r}_1 &= \frac{2d-1}{2d} N r_1 \left[-\frac{1}{(2d-1)N} + r_0 - 2r_1 + r_2\right] \\ \dot{r}_i &= \frac{2d-1}{2d} N r_1 [r_{i-1} - 2r_i + r_{i+1}], \qquad i = 2, \ldots, M-1. \end{aligned} \tag{9}$$

The last equation looks different for even and odd N:

$$\dot{r}_M = \frac{2d-1}{2d} N r_1 (2r_{M-1} - 2r_M) \qquad N = 2M \tag{10}$$

$$\dot{r}_M = \frac{2d-1}{2d} N r_1 (r_{M-1} - r_M) \qquad N = 2M + 1. \tag{11}$$

We have two stationary solutions. The first is

$$\bar{r}_1 = \bar{r}_2 = \cdots = \bar{r}_M = \bar{r}_0 - \frac{1}{(2d-1)N} \tag{12}$$

which together with the normalization condition yields

$$\bar{r}_0 = \frac{2d-2}{(2d-1)N^2} + \frac{1}{(2d-1)N} \tag{13}$$

$$\bar{r}_i = \frac{(2d-2)}{(2d-1)N^2} \qquad \text{for } i = 1, \ldots, N-1. \tag{14}$$

This solution describes the reactive steady state. Note that $\bar{r}_i \propto (d-1)$, implying a drastic difference between 1D and higher dimensional systems. In 1D, $\bar{r}_i = 0$ corresponding to coarsening is feasible, while for $d > 1$ we have $\bar{r}_1 > 0$ implying to a reactive steady state. The second stationary solution

$$\bar{r}_1 = 0 \qquad \bar{r}_i \neq 0 \qquad \text{when } i \neq 1 \tag{15}$$

corresponds to fixation; it is possible in arbitrary dimension.

To figure out which of these two solutions actually appears in the long-time limit let us solve equations (9). To accomplish this we first replace variables t and $r_j(t)$ by

$$\tau = \frac{(2d-1)N}{2d} \int_0^t \mathrm{d}t'\, r_1(t') \tag{16}$$

and

$$R_0(\tau) = r_0(t) - \frac{1}{(2d-1)N} \qquad R_i(\tau) = r_i(t). \tag{17}$$

In these variables, equations (9) acquire a pure diffusion form

$$\frac{\mathrm{d}R_j}{\mathrm{d}\tau} = R_{j-1} - 2R_j + R_{j+1}. \tag{18}$$

In these equations the index is defined modulo N as previously defined. Equivalently, we may treat $R_j(\tau)$ as a periodic function of j. The initial condition reads

$$R_j(0) = \begin{cases} \dfrac{1}{N^2} - \dfrac{1}{(2d-1)N} & j \equiv 0 \pmod N \\ \dfrac{1}{N^2} & \text{otherwise.} \end{cases} \tag{19}$$

Solving (18) subject to (19) yields

$$R_i(\tau) = \frac{1}{N^2} - \frac{1}{(2d-1)N} \sum_{j=-\infty}^{\infty} \mathrm{e}^{-2\tau} I_{i+Nj}(2\tau) \tag{20}$$

where I_n denotes the modified Bessel function of order n. If the variable $R_1(\tau) = r_1(t)$ remains positive, the modified time variable τ behaves similarly to the original time variable t; in particular, $\bar{r}_i = r_i(t = \infty) = R_i(\tau = \infty)$. The latter quantity is easily found (from the general properties of the diffusion equation) to be equal to the averaged initial value. Thus $R_i(\infty) = \frac{2d-2}{(2d-1)N^2}$, and therefore we recover the reactive steady state of equation (13). On the other hand, if $R_1(\tau)$ becomes equal to zero at some moment τ_f, this will be the end of evolution as $\tau = \tau_f$ would imply $t = \infty$. This case thus corresponds to fixation: $\bar{r}_1 = 0$, $\bar{r}_i = R_i(\tau_f) > 0$ for other i.

Practically, it is convenient to determine the minimum of $R_1(\tau)$ in the range $0 < \tau < \infty$; if the minimum is negative, fixation does happen. It turns out that the minimum becomes negative for sufficiently large N. This allows us to keep only the dominant term from the infinite sum (20), so

$$R_1(\tau) = \frac{1}{N^2} - \frac{\mathrm{e}^{-2\tau} I_1(2\tau)}{(2d-1)N}. \tag{21}$$

The minimum is reached at $\tau = \tau_* \cong 0.772\,563\,63$, and $R_1(\tau_*)$ becomes negative when $N \geqslant 4.564\,293 \times (2d-1)$. Given that N is an integer implies $N_c = 14$ in 2D. If we kept all terms in the sum, we would obtain a smaller non-integer threshold but still the same $N_c(2) = 14$. This assertion can be checked numerically with greater accuracy if we note that the sum in (20) can be significantly simplified. Indeed, using the well known identity [16]

$$\sum_{j=-\infty}^{\infty} z^j I_j(2\tau) = \exp[(z + z^{-1})\tau] \tag{22}$$

one can derive

$$\sum_{j=-\infty}^{\infty} I_{1+Nj}(2\tau) = \frac{1}{N} \sum_{p=0}^{N-1} \zeta^{-p} \exp[(\zeta^p + \zeta^{-p})\tau] \tag{23}$$

with $\zeta = \exp(2\pi\sqrt{-1}/N)$. Combining (20) and (23) we arrive at

$$R_1(\tau) = \frac{1}{N^2} - \frac{1}{(2d-1)N^2} \sum_{p=0}^{N-1} \frac{\exp[(\zeta^p + \zeta^{-p} - 2)\tau]}{\zeta^p}$$

which involves only finite summation. This expression has been used to check that indeed $R_1(\tau)$ remains positive only for $N < 14$ in 2D. One can compute $N_c(d)$ in arbitrary dimension; for instance $N_c = 5, 14, 23, 32, 42, 51$ when $d = 1, 2, 3, 4, 5, 6$, respectively.

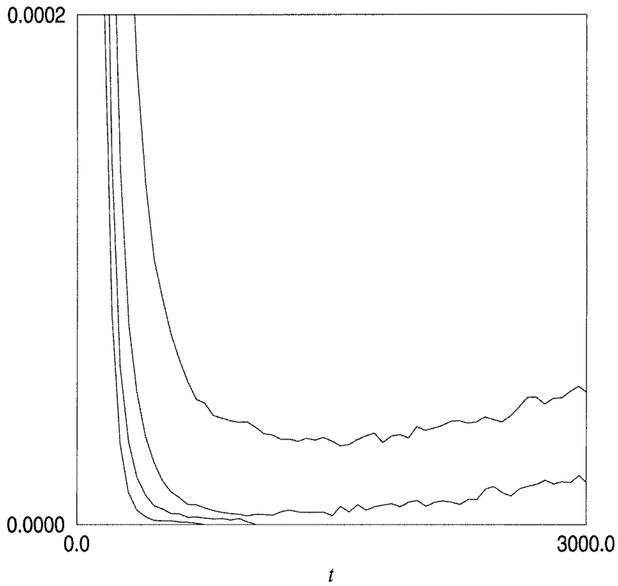

Figure 1. Time dependence of the concentrationof reactive pairs $r_1(t)$ in 2D. Shown are Monte Carlo simulation results for $N = 12, 13, 14, 15$ (top to bottom).

Thus we have found the critical number of opinions $N_c(d)$ within the realm of the Kirkwood approximation. To determine actual N_c, numerical simulations have been performed. We have considered 2D square lattices (maximum size 2048×2048) with periodic boundary conditions. On lattices of these sizes, fixation has been found for $N \geqslant 14$. The concentration of reactive pairs $r_1(t)$ is shown in figure 1 for $N = 12, 13, 14, 15$ (the simulation data represent an average of over 20 different realizations). Figure 2 plots the concentration of reactive pairs provided by the Kirkwood approximation.

A word of caution is in order. For large N, the concentration of reactive pairs saturates at a very small value. Statistical fluctuations around this value may drive this value to zero, which is an absorbing state. This would imply an apparent fixation. Even for sufficiently large systems a few samples with $N = 13$ have reached this absorbing state. However, the role of fluctuations reduces with size, and for linear sizes of order 256 and higher we have typically seen a reactive steady state when $N = 13$. In contrast, fixation has always been observed for $N = 14$ for linear sizes up to 2048. Strictly speaking, our numerical results provide a numerical lower bound for the threshold value: $N_c \geqslant 14$. However, present data support the much stronger assertion $N_c = 14$, identical to our theoretical prediction based on the Kirkwood approximation.

To demonstrate the validity of the Kirkwood approximation it is instructive to apply it to the cyclic voter model in 1D where a variety of results were already established [6, 8, 11]. For $N = 3$ and $d = 1$ we solve rate equations to find

$$r_1(t) = r_2(t) = \frac{1}{9}\frac{1}{1+t/2}. \tag{24}$$

Similarly, for $N = 4$ and $d = 1$ we find

$$r_1(t) = r_3(t) = \frac{1}{16}\frac{1}{1+t/2} \qquad r_2(t) = \frac{1}{8}\frac{1}{\sqrt{1+t/2}} - \frac{1}{16}\frac{1}{1+t/2}. \tag{25}$$

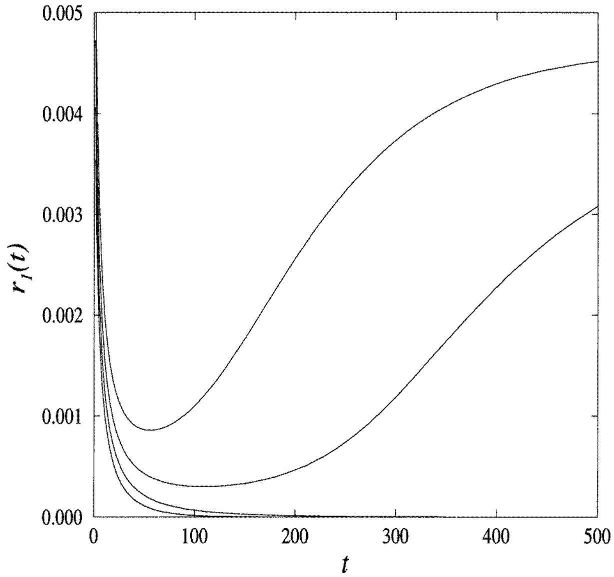

Figure 2. Numerical integration of equations (9) in 2D. Shown are the concentrations of reactive pairs for $N = 12, 13, 14, 15$ (top to bottom).

Thus in both cases the Kirkwood approximation predicts $1/t$ decay of the density of reactive interfaces. The long-time behaviours for $N = 3$ and $N = 4$ cyclic voter model in 1D agree with our previous mean-field results for these cases [11]. Compared with exact results [11], however, mean-field treatments predict faster kinetics, e.g. the density of reactive interfaces decays as $t^{-1/2}$ and $t^{-2/3}$ for $N = 3$ and 4, respectively [11]. As for the threshold number, both rigorous approaches and mean-field treatments give the same value $N_c(1) = 5$. This suggests that $N_c(d)$ given by the Kirkwood approximation might also be exact in higher dimensions.

In summary, we investigated the cyclic lattice Lotka–Volterra model. We argued that for a sufficiently large number of species, $N \geqslant N_c$, fixation occurs. Within the framework of the Kirkwood approximation, the threshold value $N_c(d)$ has been found analytically in arbitrary dimension; for instance, $N_c = 5, 14, 23, 32, 42, 51$ when $d = 1, 2, 3, 4, 5, 6$. In 1D this prediction is exact and in 2D it agrees with our numerical findings for lattices of size up to 2048×2048.

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